On-off intermittency in random map lattices driven by fractal noise

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In this paper we study numerically an ensemble of one-parameter maps driven by fractal noises. There are two accessible cases of on-off intermittency in the system for different values of parameters. The first one, corresponding to the loss of stability of the fixed point, has a power law laminar phase distribution of exponent $H-2$, while the second one, due to the instability of the synchronous motion of the ensemble, has a distribution depending on the form of the map. A simple analysis of the mechanism of this difference is also given. The distribution for the case far from the critical state is reported. [S1063-651X(96)09210-0]

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I. INTRODUCTION

Since the discovery of chaos in the Lorenz model, mechanisms of chaos in deterministic system have been carefully studied. It is well known that external noise is not necessary for the occurrence of chaos. But fluctuations, such as thermal or quantum noise, are intrinsic for real systems. So the study of random systems is of importance. Recently, a new mechanism of intermittency for random systems has attracted much interest [1–6]. It is the ''random'' switch of two states. One is a nearly constant state, named the ''off'' state; the other is a large amplitude burst, named the ''on'' state. This was first reported by Platt, Spiegel and Tresser in a set of coupled differential equations. A simpler model of a one-parameter random driving map is studied in detail by Heagy, Piatts and Hammel. Analytical results of the asymptotic $-3/2$ power-law distribution of the laminar phase is given there. On the other hand, Yu, Ott, and Chen [2] have studied a class of two-dimensional maps with randomly varying parameters. During the iteration, the size of an attractor can undergo a form of intermittency behavior that is similar to on-off intermittency. Yang and Ding found that in a random driving uncoupled map lattice with the local function of a logistic map, both phenomena can be observed [4]. Identical scaling relations imply that they are just two manifestations of on-off intermittency. All these works are for normal random driving. Ding and Yang [6] considered the system driven by the noise generated by a fractal Brownian motion (FBM) or the so-called anomalous diffusion [8–11]. The laminar phases had a power-law distribution of exponent $H-2$, where H is the Hurst exponent satisfying $0<H<1$ [9]. The past work on random driving systems is just the particular case of $H=1/2$.

In the present paper, it is shown that, for the system driven by fractal noise generated by FBM, the laminar phase distribution of the two manifestations of on-off intermittency has a different behavior. One is map dependent, while the other is not. And it is reported that for the case far from the critical state the laminar phases still have a perfect power-law distribution. The frame of the present paper is as follows: In Sec. II we give the model studied and show the numerical results of different behaviors of the two manifestations. In Sec. III the system is linearized near the two zero points of the Lyapunov exponent. A simple analysis of the difference is given there. Section IV is a short summary.

II. NUMERICAL RESULTS

A. Model

The maps we study are of the form

$$y_{n+1}^{(i)}=z_n f(y_n^{(i)}), \tag{2.1}$$

where $i=1,2,\ldots,L$, labels the ith particle (i.e., initial condition), and

$$f(y)=ye^{-by}, \tag{2.2}$$

$$z_n=ax_n, \tag{2.3}$$

with x_n a discrete-time noise generated by a FBM. A total number L of initial conditions in interval [0,1] with uniform distribution are taken. And the control parameters z_n at certain steps of iteration are the same for every initial condition. It means that all the initial conditions iterate with the same function.

Here we use the one-dimensional map below to get a deterministic FBM [6]

$$g(R)=\begin{cases} R-1+a(R-m)^z, & m\leqslant R\leqslant m+1/2 \\ R+1-a(m+1-R)^z, & m+1/2\leqslant R\leqslant m+1, \end{cases} \tag{2.4}$$

where m is an integer and exponent z specifies the properties of the diffusion. The mean square displacement $\langle R_n^2\rangle$ is expressed as [9]

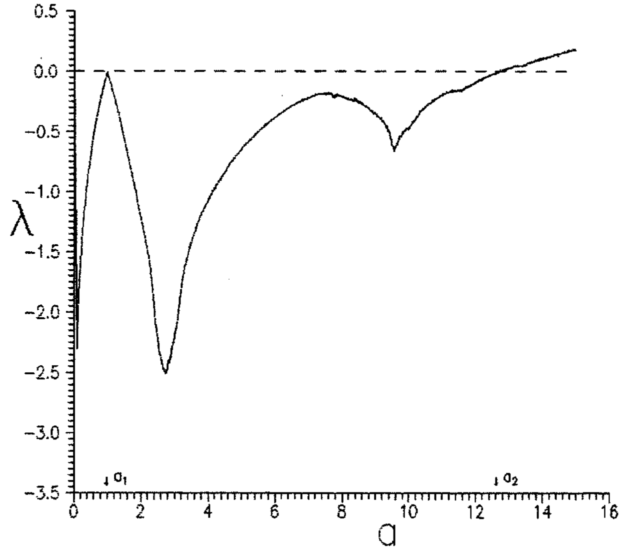

FIG. 1. The Lyapunov exponent λ for a single map with $0<a<15$, the two zero points are $a_1=1.0$ and $a_2=12.75$.

$$\langle R_n^2\rangle \sim \begin{cases} n^2 & z\geqslant 2, \quad n^{3-1/(z-1)} \\ 3/2<z<2 & n \qquad 1<z<3/2. \end{cases} \tag{2.5}$$

So the Hurst exponent is

$$H=\begin{cases} 1 & z\geqslant 2 \quad \dfrac{3}{2}-\dfrac{1}{2(z-1)} \\ 3/2<z<2 & 1/2 \qquad 1<z<3/2. \end{cases} \tag{2.6}$$

The noise x_n in Eq. (3) is

$$x_n=e^{R_n-R_{n-1}}. \tag{2.7}$$

And throughout this paper, the parameters take the values $b=2.0$, $z=1.6$. Thus we have a driving fractal noise of Hurst exponent $H=2/3$.

B. Laminar phase distribution

Chaotic or periodic motion in a single map is characterized by a positive or negative Lyapunov exponent defined by [2]

$$\lambda=\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\ln|f'(y_n)|. \tag{2.8}$$

For uncoupled map lattices, the Lyapunov exponents are the same for all the sites. Numerical calculation for a single map shows that (see Fig. 1) with increasing values of a, the Lyapunov exponent λ increases gradually to zero, and then decreases to a minimum, but it will eventually increase beyond zero. So there are two zero points for the Lyapunov exponent corresponding to two critical values $a_1\simeq 1.0$ and $a_2\simeq 12.75$. The two critical points of a divide the a axis into three intervals. Numerical simulation shows that in each interval there is a state that is different from those in the other two.

The first critical point (with $a=a_1$) is just the onset value for ''on-off intermittency''of signal y_n. This phenomenon can be observed even for a single map as studied in Refs. [1,6]. However, for different initial conditions even with values of a beyond this critical point, the motion of a large number of particles in the system are always synchronous, though the motion of a single site is random. The negative Lyapunov exponents imply that after some time all the particles must clump at a single point. From the analytical study in Ref. [6], the laminar phases show a power-law distribution with an exponential decay at large size

$$\Lambda_y(n)\propto n^{H-2}\exp(-n/n_y). \tag{2.9}$$

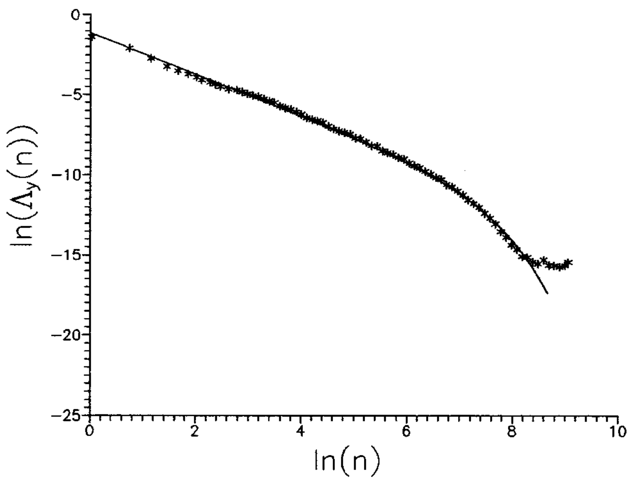

FIG. 2. The asymptotic distribution $\Lambda_y(n)$ of laminar phases with $a=1.01$. The former 100 000 steps of iteration is cut down. A total of 1 000 000 intervals of laminar phase are used to construct the distribution. The solid curve is the numerical calculation of Eq. (9). The cutoff of exponential decay is $n_y\simeq 1150$.

The numerical study for $a=1.01$ is shown in Fig. 2. The threshold from the laminar phase is fixed at $\tau=0.1$. The solid curve is a numerical calculation of Eq.(9) with Hurst exponent $H=2/3$ and a cutoff $n_y=1160$. The agreement between them is obvious.

Near the other critical point $a=a_2$, the behavior of the particle distribution is similar to that of Refs. [2,4]. For example, with $a=12.75$, we may obtain a snapshot attractor by sprinkling a large number of initial points uniformly in the interval [0,1], then iterating each point with the map (1) for a large number of iterations. The size of the snapshot attractor s_n at time n is defined [2]

$$s_n=\left[\frac{1}{L}\sum_{i=1}^{L}(y_n^{(i)}-\overline{y_n})^2\right]^{1/2}, \tag{2.10}$$

where $\overline{y_n}$ is the average of $y_n^{(i)}$

$$\overline{y_n}=\frac{1}{L}\sum_{i=1}^{L}y_n^{(i)}. \tag{2.11}$$

Just beyond the critical value a_2, the intermittency of the signal s_n is observed. Figure 3 shows the intermittency of s_n on a linear scale. It can be seen that s_n is truly of an on-off nature. Figure 4 shows the plot of laminar phases distribution from the numerical simulation of map (1) with $a=12.75$. A total of 1 000 000 intervals of laminar phase are used to con-

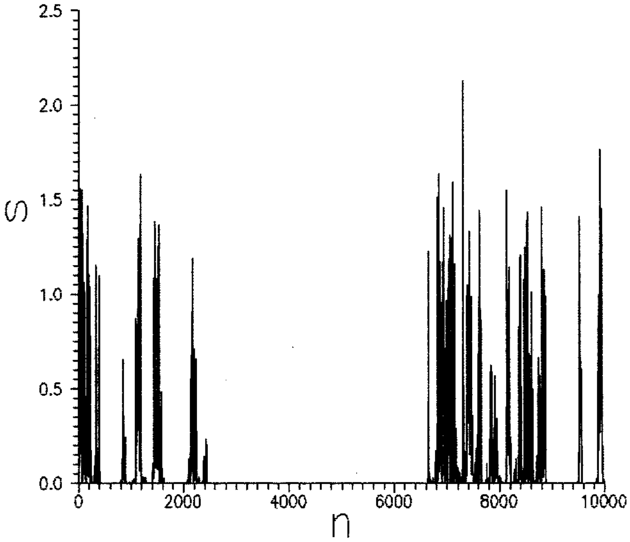

FIG. 3. The size of the snapshot s_n vs the iterate steps n on a linear scale with $a=12.75$. The dashed line is the threshold $\tau=0.01$.

struct the distribution. The threshold from a laminar phase is fixed at $\tau=10^{-2}$. The distribution is asymptotically of the power-law form. But the numerical fit gave an exponent -1.5, not the expected one $H-2$ [6].

Occasionally supercritical distribution of laminar phases was obtained. For $a=13.75$, far from the critical state, the numerical result is shown in Fig. 5. The threshold from a laminar phase is also $\tau=10^{-2}$. A total of 100 000 intervals of laminar phases were used to construct the distribution. Although the system is far from the critical state, i.e., the state with a zero-value Lyapunov exponent, the laminar phases still have a perfect power-law distribution. And the numerical fit shows that the exponent is -2.35. It is obvious that this can not be explained by the $H-2$ theory of Ding and Yang [6] because the exponent is not in the interval [-2,-1].

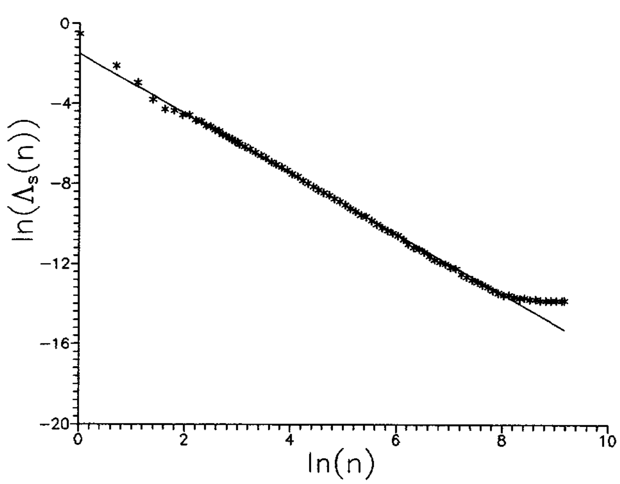

FIG. 4. The asymptotic distribution $\Lambda_s(n)$ of laminar phases with $a=12.75$. The former cutoff is 100 100 steps. The distribution is constructed by 100 000 intervals of laminar phase. The solid line is of slope -1.5.

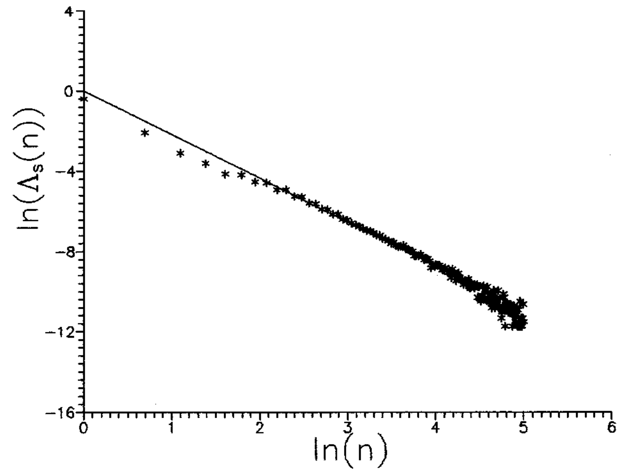

FIG. 5. The asymptotic distribution $\Lambda_s(n)$ of laminar phases with $a=13.75$. The former cutoff is 100 000 iteration. And 100 000 intervals of laminar phase are used. The solid line is of slope -2.35.

And it should also be pointed out that even the distribution for $a=12.75$ has a deviation from power law for the short length laminar phases. Much careful investigation (see Fig. 6) showed that for short length laminar phases it is still of power-law function form but with an exponent -2.35. It should be noticed that this is just the same exponent for the super-critical case $a=13.75$.

III. DISCUSSION

In this section we will analyze why the distribution near the second critical point is not of exponent $H-2$.

Near the first critical point $a_1=1.0$, the system stays at laminar phase for long periods of time. The value of y is so small at laminar phase that the map can be linearized [1,6]

$$y_{n+1}=z_n y_n \tag{3.1}$$

and taking the logrithm gives

$$\ln y_{n+1}=\ln z_n+\ln y_n . \tag{3.2}$$

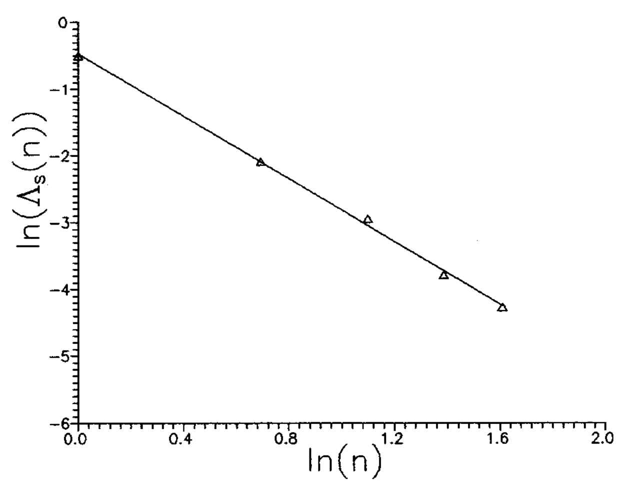

FIG. 6. The enlargement of part of Fig. 4. The solid line is of slope -2.35.

Substituting Eq. (3) and Eq. (7) into Eq. (13) and using that $\ln a_1 = 0.0$, one obtains

$$\ln y_{n+1} - \ln y_n = R_n - R_{n-1}. \tag{3.3}$$

If one regards $\ln y_i$ as the ith position of a random walker, the walk is just a FBM of the same Hurst exponent with $\{R_n\}$. So the laminar phase distribution is only determined by H but does not depend on the form of map (1).

For the case of $a_2 = 12.75$, for simplicity only two sites were considered

$$y_{n+1}^{(1)} = z_n f(y_n^{(1)}), \quad y_{n+1}^{(2)} = z_n f(y_n^{(2)}). \tag{3.4}$$

Denoting the difference between the two sites as $\delta y_n = y_n^{(1)} - y_n^{(2)}$ employing the Taylor expansion, one gets

$$\delta y_{n+1} = z_n \delta y_n f'(y_n) + O((\delta y_n)^2), \tag{3.5}$$

whose natural logarithm is

$$\ln \delta y_{n+1} = \ln z_n + \ln \delta y_n + \ln f'(y_n). \tag{3.6}$$

This equation also describes an additive random walk. But comparing it with Eq. (14), it can be seen that the extra term $\ln f'(y_n)$ in Eq. (17) is a function of y_n, i.e., it is map dependent.

Denoting

$$S_n^{(1)} = \sum_{i=0}^{n} \ln z_i, \tag{3.7}$$

$$S_n^{(2)} = \sum_{i=0}^{n} \ln f'(y_i), \tag{3.8}$$

and

$$S_n = \ln \delta y_{n+1} - \ln \delta y_0, \tag{3.9}$$

one obtains

$$S_n = S_n^{(1)} + S_n^{(2)}. \tag{3.10}$$

From Eqs. (5) and (14) and Eq. (18) one has

$$\langle S_n^{(1)} S_n^{(1)} \rangle \sim n^{2H}. \tag{3.11}$$

Assuming that

$$\langle S_n^{(2)} S_n^{(2)} \rangle \sim n^{2H'}, \tag{3.12}$$

one obtains

$$\langle S_n^2 \rangle \sim c_1 n^{2H} + c_2 n^{2H'} = \begin{cases} n^{2H} & H > H' \\ n^{2H'} & H < H', \end{cases} \tag{3.13}$$

where c_1 and c_2 are two constants. From this equation, we know that depending on the map studied the laminar phases distribution may either be of exponent $H-2$ or not. Thus we call this case a map-dependent one.

For a chaotic system, the last term acts as a chaotic walk. And from the short time memory effect of a chaotic system, we guess that the long time behavior of a chaotic walk can be viewed as a normal random walk of exponent $H = 1/2$, so we get the $-3/2$ distribution for map (1) with $a = 12.75$.

The distribution for $a = 13.75$ is somewhat similar to the case studied in Ref. [7]. It is probably the distribution of the relaxation process from the burst state to the laminar phase. A detailed analysis will be reported elsewhere.

IV. SUMMARY

A fractal noise driven random map lattice exhibiting on-off intermittency was studied in detail. The Lyapunov exponent was calculated for $0 < a < 15.0$. Near the two zero points of the Lyapunov exponent, two manifestations of on-off intermittency appeared. The laminar phase distributions for two manifestations were different; one is map dependent, and the other is not. And the power-law distribution for a far from critical case is reported. We presume that, for random modulated systems, whether described by a discrete map or by a differential equation [1–6], the Lyapunov exponent spectrumlike Fig. 1 is typical. So the on-off intermittency near the zero value of the Lyapunov exponent is also typical.

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[1] N. Platt, E.A. Spigel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993); J.F. Heagy, N. Platt, and S.M. Hammel, Phys. Rev. E **49**, 1140 (1994), N. Platt, S.M. Hammel, and J.F. Heagy, Phys. Rev. Lett. **72**, 3498 (1994).
[2] Lei Yu, E. Ott, and Qi Chen, Phys. Rev. Lett. **65**, 2935 (1990); Physica D **53**, 102 (1991).
[3] A.S. Pikovskep, Phys. Lett. A **165**, 33 (1992); T. Yamada and H. Fujisaka, Prog. Theor. Phys. **76**, 582 (1986); H. Fujisaka *et al.*, *ibid.* **76**, 1198 (1986).
[4] H.L. Yang and E.J. Ding, Phys. Rev. E **50**, R3295 (1994).
[5] E. Ott and J.C. Sommerer, Phys. Lett. A **188**, 39 (1994).
[6] M. Ding and W.M. Yang, Phys. Rev. E **52**, 207 (1995).
[7] S.D. Zhang and E.J. Ding, Phys. Lett. A **203**, 83 (1995).
[8] T. Yamada and H. Fujisaka, Prog. Theor. Phys. **84**, 824 (1990); H. Fujisaka and T. Yamada, *ibid.* **90**, 529 (1993).
[9] T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984).
[10] H.E. Hurst, Trans. Am. Soc. Civil. Eng. **116**, 770 (1951).
[11] B.B. Mandelbrot, *Fractal Geometry of Nature* (Freeman, New York,1983).